HYPERSONIC FLOW PAST A BODY IN MAGNETO-HYDRODYNAMICS

(GIPERZVUKOVOE OBTEKANIE TEL
V MAGNITNOI GIDRODINAMIKE)

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Hypersonic flow is considered past a body from within which a magnetic field is excited. The field acts on the gas that becomes electrically conducting as a result of thermal ionization occurring on transition across the strong shock wave ahead of the body. In a majority of the papers appearing recently (a bibliography can be found, e.g. [1]), the behavior of the flow of a conducting fluid is studied in the vicinity of the forward critical point of a blunt body; here the effect of a strong imposed magnetic field on the general flow picture is studied. The cases of flow past bodies of rectilinear shape – the wedge and cone – are considered in detail when the magnetic field intensity vector is directed perpendicular to the surface of the body. The method of solution is based on the assumption that the perturbed zone between the body and the shock wave is narrow [2,5]. The forces acting on the body are determined. As follows from the solution, for a sufficiently strong field the magnetic drag force has the same order of magnitude as the gasdynamic force, despite the narrowness of the zone of perturbed flow on which the magnetic field acts.

It is shown that under certain conditions the flow may separate from the wall. The point of magnetic separation from the surface of the body is determined. With increase of the imposed field this point moves upstream; therefore using strong fields leads to the possibility of creating a separated region around the body which leads to an increase in the drag of the body investigated and, as might be expected, to a decrease in the heat transfer to the body.

1. Consider a body containing the source of a magnetic field that is placed in a uniform stream of compressible inviscid gas. Magneto-hydrodynamic effects develop for very high speeds of the undisturbed stream, when ionization exists behind the intense shock wave. Therefore the Mach number of the undisturbed stream is assumed equal to infinity. The electrical conductivity in the undisturbed stream is insignificantly small, and can be neglected. Behind the shock wave in the region of flow of the conducting gas the equations of magneto-hydrodynamics are valid, which are written for the cases of plane and axisymmetric flow, for which the electric field vector is identically equal to zero [3]. This leads to a simplification in writing Ohm's law and, which is essential, preserves the validity of the Bernoulli integral along stream lines, and the Umov-Poynting vector of magnetic energy transfer proves to be identically equal to zero.

As is found from consideration of the shock wave, the Bernoulli constant, despite the presence of the magnetic field, is conserved even in transition across the shock wave, having in the entire flow a value equal to that in the undisturbed stream. In other words, for plane and axisymmetric flow Bernoulli's equation retains its original form for steady flow of an ideal gas. Taking into account the preceding remarks, the equations of magneto-hydrodynamics in dimensionless form in the region behind the shock wave take the form

$$(\mathbf{v}\nabla)\mathbf{v} + \frac{1}{\rho}\nabla p = \frac{q}{\rho}[(\mathbf{v}\times\mathbf{h})\times\mathbf{h}], \qquad q = \frac{\sigma L H^2}{c^2 R_\infty V_\infty}$$

$$\operatorname{div}\rho\mathbf{v} = 0, \qquad \operatorname{div}\mathbf{h} = 0, \qquad \operatorname{rot}\mathbf{h} = R^*[\mathbf{v}\times\mathbf{h}] \qquad (1.1)$$

$$R^* = \frac{4\pi\sigma V_\infty L}{c^2}, \qquad \frac{v^2}{2} + \frac{\varkappa}{\varkappa-1}\frac{p}{\rho} = \frac{1}{2}$$

Here $\mathbf{v}$, $\mathbf{h}$, ρ and p are respectively the velocity vector, magnetic field vector, density and pressure, κ is the ratio of specific heats, c is the speed of light in a vacuum and σ is the specific electrical conductivity, depending in the general case on temperature and pressure. Characteristic magnitudes are V_∞, H^*, R_∞, $R_\infty V_\infty^2$, which are respectively the speed of the undisturbed stream, the intensity at some point in the field, the density, and twice the dynamic pressure in the undisturbed stream. The space coordinates are referred to a characteristic length, L.

Two dimensionless quantities appear in equations (1.1), the magnetic Reynolds number, R^*, and the parameter, q, equal in order of magnitude to the ratio of magnetic to hydrodynamic forces. From estimates [4] it follows that even for very high airplane speeds (6-8 km/sec), R^* takes values less than unity. Therefore the regime of flow is considered hereafter for which the magnetic Reynolds number does not exceed the order of magnitude of one. As for the parameter q, for such speeds and for values of the imposed magnetic field of several kilogauss and airplane altitudes above 40-50 km, calculations show that q attains and may exceed a value equal to tens.

The proposed solution is constructed for the case of strong influence of the magnetic field upon the flow, when q is a large number (in actual cases of the order of ten). The formulation of the problem of flow in the absence of the shock wave is given in [3].

It follows that equations (1.1) are to be solved in conjunction with the equations for the magnetic field valid in the undisturbed stream and inside the body:

$$\operatorname{div} \mathbf{h} = 0, \qquad \operatorname{rot} \mathbf{h} = 0 \tag{1.2}$$

taking into account the presence of singularities of the magnetic field (currents creating the field) inside the body, boundary conditions at the body (the conditions of no normal flow and of continuity of the normal and tangential components of the field), and also considering the conditions at the shock wave and at infinity. The shock wave must be considered in some detail. Strictly speaking, the presence of the shock wave, which raises no doubts in the absence of the magnetic field, requires demonstration in the present case.

Actually, the conductivity of the stream behind the shock wave is finite, dissipation of magnetic energy arises, and the magneto-hydrodynamic shock wave should therefore have finite thickness in view of the dissipative factors. However, an analysis of the structure of magneto-hydrodynamic shock waves, in which the field interacts with the stream only downstream as the gas becomes sufficiently heated and therefore ionized, shows the following. At first in a length of order $1/R^0$ (R^0 is the ordinary Reynolds number) the gas dynamic quantities change, while the magnetic field remains continuous. Since viscous terms were neglected in the equations, this shock is naturally regarded as occurring instantaneously. This discontinuity in gasdynamic quantities is subject to the known relations for a shock wave in the absence of a magnetic field. After this, in a distance of the order of $1/R^*$ is developed an interaction of the stream and the magnetic field according to the laws of magneto-hydrodynamic shock waves. This situation is analogous to that encountered in a consideration of a shock wave with slow flow relaxation processes (induced by vibrational degrees of freedom of the molecule, dissociation, etc.).

With $R^* \sim 1$ the thickness of the magneto-hydrodynamic wave appears equal in order of magnitude to the characteristic dimension of the body, so that the concept of a thin wave loses its significance. Only the forward edge of this shock wave has significance – the original shock wave which separates the disturbed and undisturbed streams.

Thus, follows the assertion made previously of the conservation of the Bernoulli constant on transition across the shock wave, since this transition, despite the presence of the field, takes place according to the ordinary laws for the hydrodynamics of an ideal gas in the absence ideal gas in the absence of a field.

The position of the shock wave, different from that occupied by it in the absence of a field, must be found from the solution of the problem. The conditions behind the shock wave front (indices 1 and 2 are used respectively for conditions ahead of and behind the shock wave), expressed as in (1.1) in dimensionless variables and taking the Mach number in the undisturbed stream to be infinitely large, have the form

$$p_2 = \frac{2}{\varkappa+1}\sin^2\beta, \qquad \rho_2 = \frac{\varkappa+1}{\varkappa-1} \tag{1.3}$$
$$v_{2\tau} = \cos\beta, \quad v_{2n} = \frac{\varkappa-1}{\varkappa+1}\sin\beta, \quad \mathbf{h}_2 = \mathbf{h}_1$$

Here β is the local angle of inclination of the shock to the direction of the velocity vector of the undisturbed stream, and indices n and τ denote vector components respectively normal and tangential to the shock wave.

Equations (1.1) and (1.3) together with the conditions on the body and at infinity constitute the complete system of equations for the problem. The method of solution is based on the assumption that the perturbed zone between the shock wave and the body is narrow which, as is well known, is equivalent to the assumption that the quantity $\epsilon = (\kappa - 1)/(\kappa + 1)$ is small compared with unity. This method was used earlier by Chernyl [2] and a number of other authors (see the survey [5]. In order to exclude the inherent difficulty of this method, associated with the appearance of a point of cavitation on the surface of a curvilinear body [5], the flow is considered past bodies of rectilinear shape – the wedge and cone – where this method is known to give good results in the absence of a field. The essential simplification introduced in carrying out this method is the following. The intensity $\mathbf{h}$ of the magnetic field may be written in the form $\mathbf{h} = \mathbf{h}_0 + \mathbf{h}'$ where $\mathbf{h}_0$ is the intensity measured in the absence of the flow, satisfying equations (1.2) everywhere, and $\mathbf{h}'$ is the intensity of the induced field, satisfying the relations

$$\begin{aligned} &\operatorname{div}\mathbf{h}' = 0, \qquad \operatorname{rot}\mathbf{h}' = R^*(\mathbf{v}\times\mathbf{h}) \qquad \text{in region } II \\ &\operatorname{div}\mathbf{h}' = 0, \qquad \operatorname{rot}\mathbf{h}' = 0 \qquad\qquad\quad \text{in regions } I,\ III \end{aligned} \tag{1.4}$$

Regions *I, II* and *III* are shown in Fig. 1 where a wedge (or cone) is represented with flow in the direction of its axis. The curve L_1 represents the shock wave. Since flow is considered past a body of finite size, region *II* where the conductivity is substantially different from zero has a dimension of the order of the length of the body; immediately behind the body the gas expands, its temperature approaches the temperature of the undisturbed stream and, consequently, the conductivity returns to zero. Therefore the space behind the body is referred to as region *I*. For

$\mathbf{h}'$ the following relations can be obtained from equations (1.4):

$$\mathbf{h}'(\mathbf{r}) = \frac{\mathbf{R}^*}{4\pi}\,\mathrm{rot} \iiint\limits_{\tau} \frac{\mathbf{v}(\mathbf{R}) \times \mathbf{h}(\mathbf{R})}{|\mathbf{R}-\mathbf{r}|}\, d\tau \quad \text{(for three-dimensional flow)}$$
$$\mathbf{h}'(\mathbf{r}) = \frac{\mathbf{R}^*}{2\pi}\,\mathrm{rot} \iint\limits_{S} [\mathbf{v}(\mathbf{R}) \times \mathbf{h}(\mathbf{R})] \ln \frac{1}{|\mathbf{R}-\mathbf{r}|}\, dS \quad \text{(for plane flow)} \qquad (1.5)$$

Here $\mathbf{r}$ is the radius vector of a point at which the magnetic intensity is determined, $\mathbf{R}$ is a variable radius vector, and τ is the volume (S correspondingly the area) of disturbed flow (region II in Fig. 1). The dimension of τ or S along the surface of the body is of order unity in dimensionless variables, the thickness of the zone is of order ϵ, and the integrand in relations (1.5) is of order unity. Hence $h' \sim \epsilon$, which permits the vector $\mathbf{h}$ to be replaced by the vector $\mathbf{h}_0$ in equation (1.5), from which a closed expression is obtained for finding the induced field. Then the circumstance that the induced field is of order ϵ permits the vector $\mathbf{h}$ to be replaced by the given vector $\mathbf{h}_0$ in the equations of motion of the gas with acceptable accuracy. Thus the formulation of the problem is changed. The problems for finding the hydrodynamic and induced magnetic fields, which are found together in the general case, are separated in the new formulation, which renders the solution practicable.

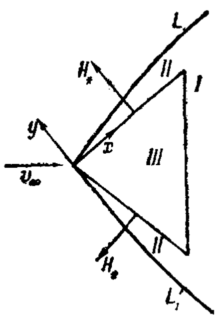

Fig. 1.

2. Flow past a wedge. Equations (1.1) in a Cartesian coordinate system have the following form

$$\begin{aligned}
&u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{1}{\rho}\frac{\partial p}{\partial x} = (vh_x h_y - uh_y^2)\frac{q}{\rho}\\
&u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \frac{1}{\rho}\frac{\partial p}{\partial y} = (uh_x h_y - vh_x^2)\frac{q}{\rho}\\
&\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} = 0, \qquad \frac{\partial h_x}{\partial x} + \frac{\partial h_y}{\partial y} = 0\\
&\frac{\partial h_y}{\partial x} - \frac{\partial h_x}{\partial y} = R^*(uh_y - vh_x), \quad \frac{v^2}{2} + \frac{u^2}{2} + \frac{\varkappa}{\varkappa - 1}\frac{p}{\rho} = \frac{1}{2}
\end{aligned} \qquad (2.1)$$

The x-axis is directed along the surface of the body, the origin of coordinates coincides with the vertex of the wedge and u and v are the components of the velocity vector along the x- and y-axis respectively (Fig. 1).

It is assumed henceforth that the magnetic field vector is directed along the y-axis and has a constant modulus H^*, chosen as the characteristic quantity for the magnetic field.

The length of the surface of the wedge is chosen as the characteristic length L. We estimate the quantities appearing in equations (2.1). The case is considered when a strong magnetic field acts: $q \sim \epsilon^{-1}$. Taking into account the ideas advanced in Section 1 regarding the induced magnetic field, we have for the components of the magnetic field intensity vector **h**

$$h_x \sim \varepsilon, \ h_y = 1 + O(\varepsilon)$$

For x, y, u, v, ρ we have as in the case of no magnetic field

$$x \sim 1, \quad y \sim \varepsilon, \quad u \sim 1, \quad v \sim \varepsilon, \quad \rho \sim \varepsilon^{-1}$$

From the second of equations (2.1) (projection of forces on the y-axis) it follows that $\partial p/\partial y = O(1)$, from which $p_2 - p = O(\epsilon)$, where the difference between the pressure p_2 behind the shock wave and the pressure p at any point in the disturbed flow is taken for fixed x.

The equation of the shock wave is written $y = Y(x)$, where $Y(x) \sim \epsilon$. Hence, according to equation (1.3), it follows that $p_2 = \sin^2\theta + O(\epsilon)$, where θ is the semi-vertex angle of the wedge. Consequently, the pressure in the whole disturbed flow region is constant to within an accuracy of ϵ, from which $\partial p/\partial x \sim \epsilon$ which is used in the first of the equations of motion (2.1). Finally, meglecting magnitudes ϵ in comparison with unity, we obtain

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -u\frac{q}{\rho}, \qquad p = \sin^2\theta$$

$$\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} = 0, \qquad u^2 + \frac{1}{\varepsilon}\frac{p}{\rho} = 1, \qquad \frac{\partial h'_y}{\partial y} = 0, \qquad \frac{\partial h'_x}{\partial y} = -R^* u \tag{2.2}$$

The following two equations give the induced magnetic field

$$h_y' = \gamma(x), \qquad h_x' = -R^* \int_0^y u(x, \eta)\,d\eta + \delta(x) \tag{2.3}$$

where the functions $\gamma(x)$ and $\delta(x)$ can be found by turning to the expression for **h′** (1.5). Taking account of the fact that the width of the region of integration is of order ϵ, we obtain with acceptable accuracy, transforming (1.5)

$$h'_y = \frac{R^*}{2\pi}\frac{\partial}{\partial x}\int_0^1\int_0^{Y(x)} u(\xi, \eta)\ln|x - \xi|\,d\xi\,d\eta \tag{2.4}$$

$$h_x' = \frac{R^*}{2}\int_0^{Y(x)} u(x, \eta)\,d\eta - R^*\int_0^y u(x, \eta)\,d\eta$$

It is easy to see from the expression (2.4) that the zone of disturbed flow is equivalent for the magnetic field considered to a vortical layer (current sheet) of thickness of order ϵ per unit length. For the hydrodynamic field, according to (1.3) we obtain

on the shock wave

$$p_2 = \sin^2\theta, \quad \rho_2 = \frac{1}{\varepsilon}, \quad u_2 = \cos\theta, \quad v_2 = \frac{dY}{dx}\cos\theta - \varepsilon\sin\theta \quad \text{at} \quad y = Y(x) \qquad (2.5)$$

on the wedge

$$v = 0 \qquad \text{at} \quad y = 0 \qquad (2.6)$$

We introduce in place of x and y the independent variables x, ψ, taking ψ as the stream function which satisfies

$$\frac{\partial\psi}{\partial x} = \rho v, \qquad \frac{\partial\psi}{\partial y} = -\rho u, \quad \psi = 0 \quad \text{on the surface of the wedge.} \qquad (2.7)$$

Then from equations (2.2) we obtain

$$\frac{\partial u}{\partial x} = -\frac{q}{\rho}, \quad p = \sin^2\theta, \qquad u^2 + \frac{1}{\varepsilon}\frac{p}{\rho} = 1, \qquad \frac{\partial}{\partial\psi}\frac{v}{u} + \frac{\partial}{\partial x}\frac{1}{\rho u} = 0 \qquad (2.8)$$

Expressing $1/\rho = \epsilon(1 - u^2)/\sin^2\theta$ and substituting this expression into the first of equations (2.8), taking (2.5) into account, we obtain an integral for u:

$$u = \operatorname{th}\left[\ln\operatorname{ctg}\frac{\theta}{2} - \frac{\varepsilon q}{\sin^2\theta}(x - X)\right] \qquad (2.9)$$

where $x = X(\psi)$ is the equation of the shock wave in the new independent variables. Hence for the density ρ we have

$$\rho = \frac{\sin^2\theta}{\varepsilon}\operatorname{ch}^2\left[\ln\operatorname{ctg}\frac{\theta}{2} - \frac{\varepsilon q}{\sin^2\theta}(x - X)\right] \qquad (2.10)$$

The integral for u is written for the case of a constant value of the electrical conductivity σ in the entire flow region. It would not be difficult to write this integral considering variable σ. Thus, using the dependence of conductivity upon temperature $\sigma = \sigma_0(T/T_0)^n$, where σ_0 is the conductivity at a certain temperature T_0, and choosing as T_0 the stagnation temperature, we obtain the dependence of conductivity on the dimensionless speed $\sigma = \sigma_0(1 - u^2)^n$, from which the expression for u assumes the implicit form

$$x - X(\psi) = -\frac{\sin^2\theta}{\varepsilon q_0}\int_{\cos\theta}^{u}\frac{du}{(1 - u^2)^n} \qquad (2.11)$$

where σ_0 enters into the expression for q_0. The expression (2.11) is inconvenient for subsequent consideration. Therefore we take everywhere q = const, that is the integral (2.9) is used, although the analysis associated with the expression (2.11) develops in general analogous to the case q = const.

According to equation (2.9), for $x > \xi$, where

$$\xi = \frac{\sin^2\theta}{\varepsilon q} \ln \operatorname{ctg} \frac{\theta}{2} \tag{2.12}$$

the speed u on the surface of the wedge $X(\psi) = 0$ changes sign, becoming a negative quantity.

Physically this phenomenon is explained by the action of intense pondero-motive forces, directed contrary to the direction of motion. Fluid particles in the disturbed flow region are slowed down under the action of these forces until finally their speed becomes zero, after which a reverse flow arises at the surface of the body. The phenomenon connected with the appearance of reverse flow is analogous to the phenomenon of viscous separation of a boundary layer, and may by analogy be termed the phenomenon of magnetic separation, and the point ξ the magnetic separation point. For x near to ξ, the given solution is inapplicable. However, using the integral (2.9) it is possible to determine, even though only qualitatively, the behavior of the flow in the vicinity of the magnetic separation point.

In the neighborhood of $x = \xi$ we have, resolving the expression for u into a series and retaining only first order terms,

$$u = -\frac{\varepsilon q}{\sin^2\theta} (x - \xi - c\psi) \qquad \left(c = \frac{dX(\psi)}{d\psi}\bigg|_{\psi=0}\right) \tag{2.13}$$

At the separation point $\rho = \epsilon^{-1} \sin^2\theta$. Consequently, according to (2.7)

$$\frac{\partial\psi}{\partial y} = q(x - \xi - c\psi) \tag{2.14}$$

Hence, integrating and taking into account the fact that $\psi = 0$ at $y = 0$, we have

$$\psi = \frac{x-\xi}{c} e^{-qcy}(e^{qcy} - 1)$$

For small y this gives $\psi = q(x - \xi)y$.

The stream lines are depicted in Fig. 2, representing qualitatively the actual behavior of the flow near the singular point. The zone behind the separation point (region S in Fig. 3) represents, apparently, a separated region of vortical flow. The presence of this zone leads to an effective thickening of the body, and an increase in its drag.

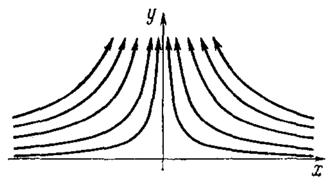

Fig. 2.

It is understood that the separated zone appears on the body only in the case $\xi < 1$. With increase of the field intensity (q increasing) the magnetic separation point moves upstream. For very strong fields the body may be almost entirely surrounded by the separated zone, which increases the drag of the body and, as might be expected, reduces the heat transfer to the body. For $\xi \ll 1$ it is proper to choose as the characteristic length not L, as was assumed everywhere until now, but the abscissa of the separation point, that is the quantity ξL. Denoting the parameter q corresponding to the new characteristic length by q_ξ, we have according to (2.12)

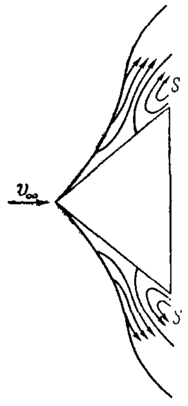

Fig. 3.

$$q_\xi = \frac{\sin^2\theta}{\varepsilon} \ln \operatorname{ctg}\frac{\theta}{2} \qquad (2.15)$$

For such a choice of the characteristic length we have always $q \sim \epsilon^{-1}$, as was assumed in constructing the solution. Henceforth the quantity ξL will always be chosen as the characteristic length. It follows that it should be emphasized that the phenomenon of magnetic separation is distinguished from the phenomen, known in the theory of an infinitely conducting gas ($R^* = \infty$), of "twisting" of the stream with the aid of a magnetic field, since in the latter case the magnetic field does not penetrate into the flow, forming on the edge of the stream as the so-called magnetic wall.

In [1], where magneto-hydrodynamic flow was considered in the vicinity of the forward critical point, the existence of a singularity in the solution was noted for certain values of the parameter q (in our notation), denoted as critical by the author. For values of q exceeding the critical, the solution loses its physical significance.

It follows that it can be expected, in accord with the investigation made in the present work, that for super-critical values of the parameter q uniform flow near the critical point is impossible. Here a separated zone appears that is analogous to the corresponding phenomenon for flow past a blunt body with "needle," with the difference only that the role of the "needle" is played by the magnetic field.

In the case considered ($R^* \sim 1$) the lines of magnetic force pierce the flow region, exerting on the stream not the force of "magnetic pressure", but rather the force of magnetic friction [3].

We find the form of the shock wave. The expression for v, according to the last of equations (2.8), has the form

$$v = -u \int_0^\psi \frac{\partial}{\partial x} \frac{1}{\rho u} d\psi \qquad (2.16)$$

Carrying out the differentiation under the integral sign, with consideration of the expressions (2.9) and (2.10), we obtain

$$v = -\frac{4\varepsilon^2 q_\xi}{\sin^4\theta}\,\mathrm{th}\,[a - b(x - X)]\int_0^{\psi}\frac{\mathrm{ch}\,[2a - 2b(x - X)]}{\mathrm{sh}^2\,[2a - 2b(x - X)]}\,d\psi \qquad (2.17)$$

where

$$a = \ln \mathrm{ctg}\,\frac{\theta}{2}, \qquad b = \frac{\varepsilon q_\xi}{\sin^2\theta}$$

The function $X(\psi)$ is subject to determination from the boundary conditions at the shock wave. Use is made of the expression for the stream function in the undisturbed stream

$$\psi = -x\sin\theta - y\cos\theta \qquad (2.18)$$

Hence, introducing the function $\psi = \Psi(x)$, the inverse of $X(\psi)$, and taking into account the continuity of the stream function at the shock wave, we have

$$\Psi(x) = -x\sin\theta - Y(x)\cos\theta \qquad (2.19)$$

Hence, according to (2.5)

$$v = -\frac{d\Psi(x)}{dx} - \sin\theta\,(1 + \varepsilon) \qquad (2.20)$$

Substituting the expression (2.20) into the left side of equation (2.17) and transforming the integral to an integration with respect to the variable $s = X(\psi)$, we obtain for the determination of $d\Psi/dx$ the linear integral equation

$$\frac{d\Psi}{dx} + \sin\theta\,(1 + \varepsilon) = \frac{4\varepsilon^2 q_\xi\cos\theta}{\sin^4\theta}\int_0^{x}\frac{\mathrm{ch}\,[2a - 2b(x - s)]}{\mathrm{sh}^2[2a - 2b(x - s)]}\,\frac{d\Psi}{ds}\,ds \qquad (2.21)$$

The kernel of this integral equation becomes infinite at $x = 1$, $s = 0$. Analysis shows that $\Psi(x)$ also becomes infinite in the vicinity of $x = 1$, having the logarithmic singularity $\Psi(x) \sim \ln|x - \xi|$.

In the vicinity of the magnetic separation point the solution (2.21) is inapplicable. The distance from this point at which the solution loses force can be judged from the solution itself which becomes useless when the shock wave lies away from the body a distance large in comparison with ϵ. The solution of equation (2.21) can be carried out by a method analogous to the method fo Euler for the solution of ordinary differential equations. It follows that the solution is carried out starting with the point $x = 0$. After the determination of $d\Psi/dx$, and then also $\Psi(x)$, the function $Y(x)$ is determined, giving the equation of the shock wave from equation (2.19). It must be noted that at the tip of the wedge the angle of inclination of the shock wave is equal to the corresponding value for the case when the magnetic field is absent:

$$\frac{dY(x)}{dx} = \varepsilon \operatorname{tg} \theta$$

In Fig. 4 are presented the results of the solution $Y(x)$ for $\theta = 40^0$ (solid line), from which it follows that in the plane case the method presented gives low accuracy: the solution ceases to be good at $x \sim 0.6$ for $\chi = 1.4$, and $x \sim 0.8$ for $\kappa = 1.2$. (The characteristic length has been chosen as $L\xi$, the abscissa of the separation point.) As will follow from what comes later, in the axisymmetric case, the accuracy of the method is considerably greater.

It is possible to obtain an analytical solution of equation (2.21) if the method of successive approximations is used: in so doing since the equation itself was obtained with an accuracy of ϵ^2, it is sufficient to take the first approximation. Taking as the zero approximation $\Psi_0(x)$, the expression for the stream function in the absence of a field, we obtain

$$\left(\frac{d\Psi}{dx}\right)_0 = -\sin\theta\,(1+\varepsilon) \tag{2.22}$$

Inserting the expression (2.22) under the integral sign in equation (2.21) and integrating twice, we obtain an expression for the first approximation $\Psi_1(x)$:

$$\Psi_1(x) = -x\sin\theta + \frac{\cos\theta\sin\theta}{q_\xi}\ln\left[\sec\theta \operatorname{th}\left(\ln\operatorname{ctg}\frac{\theta}{2} - \frac{\varepsilon q_\xi x}{\sin^2\theta}\right)\right] \tag{2.23}$$

Hence the equation for the shock wave, according to (2.19), is

$$Y(x) = \frac{\sin\theta}{q_\xi}\ln\left[\sec\theta \operatorname{th}\left(\ln\operatorname{ctg}\frac{\theta}{2} - \frac{\varepsilon q_\xi x}{\sin^2\theta}\right)\right] \tag{2.24}$$

In Fig. 4 the dashed lines give the results of calculations with this formula. From the graph it follows that the expression (2.24) gives a good approximation to the exact solution in its region of applicability.

We consider the forces acting on the body. Aside from the pressure force acting normal to the surface of the body, determined with an accuracy of order ϵ by Newton's formula, a magnetic force acts on the body that can be calculated according to [3]. However, in the present case it is convenient to use the impulse theorem, the application of which gives the drag force referred to the surface of the wedge

$$F^* = L\xi R_\infty U_\infty^2 \int_0^{Y(x)} (\cos\theta - u)\,\rho u\,dy \tag{2.25}$$

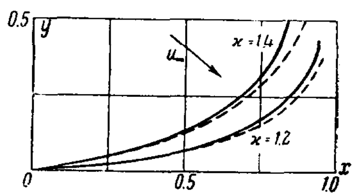

Fig. 4.

This formula determines the force as a summation of the loss of impulse resulting from the action of the magnetic forces. The integral is taken for a certain fixed x, where it is assumed that $\xi \ll 1$. Letting $x \to 1$ and transforming to an integration with respect to ψ, and then with respect to $s = X(\psi)$, we have

$$F^* = -L\xi R_\infty U_\infty^2 \lim_{x\to 1}\int_0^x \left\{\cos\theta - \operatorname{th}\left[\ln\operatorname{ctg}\frac{\theta}{2} - \frac{q_\xi}{\sin^2\theta}(x-s)\right]\right\}\frac{d\Psi}{ds}\,ds \qquad (2.26)$$

Carrying out the integration with an accuracy of ϵ, we obtain

$$F^* = L\xi R_\infty U_\infty^2 \sin\theta\left(\cos\theta + \frac{\ln\sin\theta}{\ln\operatorname{ctg}{}^1/_2\,\theta}\right) \qquad (2.27)$$

Projecting this force in the direction of motion, and comparing its projection, Q, with the hydrodynamic drag force, F, determined from Newton's formula that acts on the portion of the wedge to the separation point, we obtain $Q = 0.635F$ for $\theta = 40^\circ$. Such is the relation between the hydrodynamic and magnetic forces acting on the part of the wedge back to the separation point. It is difficult to determine the hydrodynamic and magnetic forces acting on the body as a result of the appearance of the separated region. It may be expected that the amount of drag contributed by the presence of the separated region is of the same order as the drag experienced by the body in the absence of a magnetic field.

3. Flow past a cone. The solution for the cone is constructed in general just as for the wedge; therefore the exposition of the solution will be brief. The equations of motion in a spherical system of coordinates r and ϑ, where r is the radius vector and ϑ the angle measured from the axis of the cone, have the form

$$\begin{gathered}
u\frac{\partial u}{\partial r} + \frac{v}{r}\frac{\partial u}{\partial\vartheta} - \frac{v^2}{r} + \frac{1}{\rho}\frac{\partial p}{\partial r} = (vh_r h_\vartheta - uh_\vartheta^2)\frac{q}{\rho} \\
u\frac{\partial v}{\partial r} + \frac{v}{r}\frac{\partial v}{\partial\vartheta} + \frac{uv}{r} + \frac{1}{\rho r}\frac{\partial p}{\partial\vartheta} = (uh_r h_\vartheta - vh_r^2)\frac{q}{\rho} \\
\frac{\partial r^2\rho u\sin\vartheta}{\partial r} + \frac{\partial r\rho v\sin\vartheta}{\partial\vartheta} = 0, \qquad \frac{u^2}{2} + \frac{v^2}{2} + \frac{\varkappa}{\varkappa-1}\frac{p}{\rho} = \frac{1}{2}
\end{gathered} \qquad (3.1)$$

The origin of coordinates is at the vertex of the cone, and u and v are respectively the radial and tangential components of velocity. The characteristic length will be taken to be the distance from the vertex of the cone to the magnetic separation point calculated, as will follow from what comes later, just as in the plane case from the relation (2.12). The equation for the induced magnetic field is not written down, since this field can be found from formula (1.5). Here it is also necessary to obtain, using the narrowness of the disturbed region, equations of the type (2.3) and (2.4) for the induced magnetic field.

Setting $\vartheta = \theta + \phi$, where θ is the semi-vertex angle of the cone and

$\phi \sim \epsilon$ we obtain, with just the same assumptions as in Section 2, the following equations:

$$u\frac{\partial u}{\partial r} + \frac{v}{r}\frac{\partial u}{\partial \varphi} = -u\frac{q}{\rho}, \qquad p = \sin^2\theta$$
$$u^2 + \frac{1}{\varepsilon}\frac{p}{\rho} = 1, \qquad \frac{\partial r^2\rho u \sin\theta}{\partial r} + \frac{\partial r\rho v \sin\theta}{\partial \varphi} = 0 \qquad (3.2)$$

As before, we take the imposed magnetic field perpendicular to the cone surface. The boundary conditions on the surface of the shock wave are

$$p_2 = \sin^2\theta, \qquad \rho_2 = \frac{1}{\varepsilon}, \qquad u_2 = \cos\theta, \qquad v_2 = r\frac{d\Phi}{dr}\cos\theta - \varepsilon\sin\theta \qquad (3.3)$$

where $\phi = \Phi(r)$ is the equation of the shock wave line in the spherical coordinate system. As for plane flow, independent variables r and ψ are introduced, where ψ is the stream function, determining

$$\frac{\partial\psi}{\partial r} = r\rho v\sin\theta, \qquad \frac{\partial\psi}{\partial\varphi} = -r^2\rho u\sin\theta \qquad (3.4)$$

In the new variables equation (3.2) takes the form

$$\frac{\partial u}{\partial r} = -\frac{q}{\rho}, \quad p = \sin^2\theta, \quad u^2 + \frac{p}{\varepsilon\rho} = 1, \quad \frac{\partial}{\partial\psi}\frac{v}{ur} + \frac{\partial}{\partial r}\frac{1}{r^2\rho u\sin\theta} = 0 \qquad (3.5)$$

The integral for u has the form (2.9), and the expression for ρ is (2.10), where r appears in place of x, and $R(\psi)$ instead of $X(\psi)$, the equation of the shock wave in the new independent variables.

According to (3.5) the equation for v has the form

$$v = -\frac{ru}{\sin\theta}\int_0^{\psi}\frac{\partial}{\partial r}\left[\frac{1}{r^2\rho u}\right]d\psi \qquad (3.6)$$

Substituting into this expression for ρ and u from equations (2.9) and (2.10), after a series of transormations we obtain

$$v = -\frac{4\varepsilon^2 q_\xi \,\mathrm{th}\,[a - b(r-R)]}{r\sin^5\theta}\int_0^{\psi}\frac{\mathrm{ch}\,[2a - 2b(r-R)]}{\mathrm{sh}^2\,[2a - 2b(r-R)]}d\psi +$$
$$+ \frac{4\varepsilon\,\mathrm{th}\,[a - b(r-R)]}{r^2\sin^3\theta}\int_0^{\psi}\frac{d\psi}{\mathrm{sh}\,[2a - 2b(r-R)]} \qquad (3.7)$$

where a and b are determined just as in (2.17).

We write the boundary conditions for the determination of the function $R(\psi)$. The expression for the stream function in the undisturbed stream has the form

$$\psi = -\tfrac{1}{2}r^2\sin^2\vartheta \qquad (3.8)$$

Hence on the shock wave with acceptable accuracy we have

$$\Psi(r) = -\frac{1}{2} r^2 \sin^2\theta - r^2 \sin\theta\cos\theta\Phi(r) \tag{3.9}$$

Proceeding from this, the condition for v on the shock wave can, with consideration of (3.3), be written

$$v = -\frac{r}{\sin\theta}\frac{d}{dr}\frac{\Psi}{r^2} - \varepsilon\sin\theta \tag{3.10}$$

Since for r near zero $\Psi \sim r^2$, on the shock wave at the tip of the cone we have

$$v = -\varepsilon\sin\theta \qquad \text{at } r = 0 \tag{3.11}$$

We determine the shock angle at the tip of the cone, which can be used to verify the accuracy of equation (3.7).

Letting $r \to 0$ in (3.7) we find that the first term on the right side of the expression (3.7) vanishes, and the second term gives

$$v = \frac{2\psi\varepsilon}{r^2\sin\theta} \tag{3.12}$$

On the other hand, integrating the second of equations (3.4) we obtain for r near zero

$$\Psi = -\frac{r^2}{\varepsilon}\cos\theta\sin\theta\Phi(r) \tag{3.13}$$

Finally for v on the shock wave we have the equation

$$v = -2\cos\theta\Phi(r) \tag{3.14}$$

This together with (3.11) gives an expression for the angle of inclination Φ_0 of the shock at the tip of the cone:

$$\Phi_0 = \frac{1}{2}\varepsilon \operatorname{tg}\theta \qquad \text{at } r = 0 \tag{3.15}$$

in accord with the result obtained for a cone in the absence of a magnetic field [2].

Using (3.9) and (3.15) we obtain the stream function on the shock wave in the absence of a magnetic field:

$$\Psi = -\frac{1}{2} r^2 \sin^2\theta(1+\varepsilon) \tag{3.16}$$

We now write the equation for $\Psi(r)$. Substituting the expression (3.10) into the left side of equation (3.7) and transforming to an integration with respect to $s = R(\psi)$, we obtain an integro-differential equation for the determination of the function $\Psi(r)$:

$$-\frac{r}{\sin\theta}\frac{d}{dr}\frac{\Psi}{r^2} - \varepsilon\sin\theta = -\frac{4\varepsilon^2 q_\xi \cos\theta}{r\sin^5\theta}\int_0^r \frac{\operatorname{ch}[2a-2b(r-s)]}{\operatorname{sh}^2[2a-2b(r-s)]}\frac{d\Psi}{ds}\,ds +$$

$$+\frac{4\varepsilon\cos\theta}{r^2\sin^3\theta}\int_0^r \frac{1}{\operatorname{sh}[2a-2b(r-s)]}\frac{d\Psi}{ds}\,ds \tag{3.17}$$

For the solution of equation (3.17) we apply the method of successive approximations, taking (3.16) as the zero approximation. We perform the integration on the right side of (3.17), divide equation (3.17) by r and integrate once more, taking account of

$$\lim_{r\to 0}\frac{\Psi}{r^2} = -\frac{1+\varepsilon}{2}\sin^2\theta \tag{3.18}$$

After a series of transformations we obtain the first approximation for Ψ:

$$\Psi_1(r) = -\frac{\sin^2\theta}{2}r^2 + \frac{\cos\theta\sin^2\theta}{q_\xi}\int_0^r \ln\left\{\sec\theta\,\operatorname{th}\left[\ln\operatorname{ctg}\frac{\theta}{2} - \frac{\varepsilon q_\xi}{\sin^2\theta}z\right]\right\}dz \tag{3.19}$$

It is curious to note that the expression for $\Psi_1(r)$ found for the case of a wedge (2.23) is equal to the derivative with respect to r of Ψ_1 for the cone divided by $\sin\theta$. The equation of the shock wave now takes the following form, according to (3.9):

$$\Phi(\varphi) = -\frac{\sin\theta}{q_\xi r^2}\int_0^r \ln\left\{\sec\theta\,\operatorname{th}\left[\ln\operatorname{ctg}\frac{\theta}{2} - \frac{\varepsilon q_\xi}{\sin^2\theta}z\right]\right\}dz \tag{3.20}$$

In Fig. 5 are shown the results of a calculation according to formula (3.20) of the equation of the shock wave for a cone with semi-vertex angle $\theta = 40^0$ for $\kappa = 1.4$ and $\kappa = 1.2$. In Fig. 5 the shock wave is shown in a Cartesian coordinate system, the x-axis being directed along the surface, and the y-axis perpendicular to the body surface. As is clear from the calculations, this method gives significantly better results for the axisymmetric than for the plane flow.

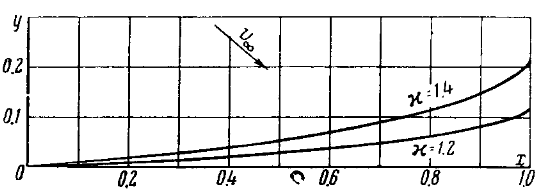

Fig. 5.

The distance, y, between the shock wave and the body, which becomes infinite in the vicinity of the magnetic separation point for the case of the wedge, has the following form in the neighborhood of $x = 1$ in the present case: $y \sim (1-x)\ln(1-x)$; that is, the derivative dy/dx becomes infinite but not the function itself. In connection with this, the solution is applicable almost up to the magnetic separation point.

The flow in the vicinity of the magnetic separation point is written analogously to the case of flow past a wedge. For the velocity we have an expression coming from the integral (2.9), where x and $X(\psi)$ are replaced respectively by r and $R(\psi)$, with consideration of the expression (3.16):

$$u = -\frac{\varepsilon q_\xi}{\sin^2\theta}(r - 1 - A\sqrt{-\psi}) \qquad \left(A = \frac{1}{\sin\theta}\sqrt{\frac{2}{1+\varepsilon}}\right) \tag{3.21}$$

From this and equation (3.4), transforming to a local Cartesian coordinate system s and y with origin at the magnetic separation point (and the y-axis directed perpendicular to the surface of the cone), we obtain

$$\frac{\partial\psi}{\partial y} = q_\xi(s - A\sqrt{-\psi})\sin\theta \tag{3.22}$$

Integrating this equation, considering that $\psi = 0$ at $y = 0$:

$$\frac{q_\xi \sin\theta}{2} y = \frac{\sqrt{-\psi}}{A} + \frac{s}{A^2}\ln\left|\frac{s - A\sqrt{-\psi}}{s}\right| \tag{3.23}$$

The stream lines according to equation (3.23) are shown in Fig. 6. It is understood that the solution is inapplicable for $s > 0$. Attention is drawn to the fact that, in contrast to the plane case (Fig. 2), in the present case the streamlines do not go off to infinity, intersecting the y-axis at a finite distance from the body surface.

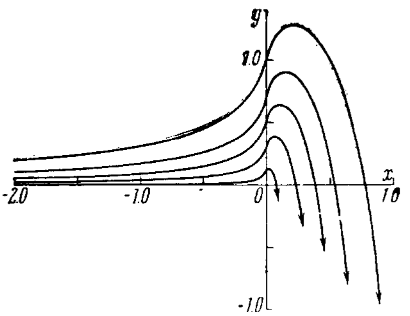

Fig. 6.

We determine the magnetic drag force acting on the portion of the cone up to the magnetic separation point. According to the impulse theorem, after a series of transformations analogous to those carried out in Section 2, we have the following expression for the force

$$Q = 2\pi\cos\theta\sin^2\theta\left[\frac{\cos\theta}{2} - \frac{1}{(\ln \operatorname{ctg} {}^1\!/_2\theta)^2}\int_0^a z \operatorname{th} z\, dz\right] R_\infty\xi^2 L^2 U_\infty^2 \tag{3.24}$$

$$(a = \ln \operatorname{ctg} {}^1\!/_2\theta)$$

Comparing this force with that found from Newton's formula for the hydrodynamic force, for $\theta = 40^\circ$ we obtain $Q = 0.40F$.

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Translated by M.D.V.D.